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On the optimal direction of short metal fibres in brittle matrix composites

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In the composite materials considered in this paper the fibres bridge the cracks in the matrix and control their propagation. The ability to sustain large cracks before complete failure is of primary importance in several applications. This quality formulated as the fracture energy is chosen as an objective function for optimization. Five different components of the fracture energy are expressed by simplified formulae, derived from the assumed behaviour of fibres in the cracked matrix. The angle of the orientation of the parallel fibres system is the only design variable. The optimization problem is solved by derivation with respect to that angle. An element subjected to axial tension is considered for the maximum of fracture energy and the optimum angle of fibres is determined. Several examples for steel fibres reinforced cements are calculated and discussed. The proposed approach may be further developed using adequate formulae and assumptions for various kinds of fibre reinforced materials.

1. Introduction

In materials engineering the internal structure and the composition of a material are designed to fulfil preassigned requirements and conditions. For brittle matrix composites, in which fibres should control crack development, the important mechanical property is the fracture toughness. This is the ability of a material to resist fracture by crack propagation. The fracture toughness is often defined as proportional to the area under the force–displacement of stress–strain diagram.

The optimization approach is aimed at objective methods of material design, according to precisely defined criteria and conditions. Already Mullin and Mazzio [1] have attracted attention in the optimization of composite behaviour during the fracture process. Such an approach seems particularly rational for cement based matrices reinforced with short steel fibres. These materials are of growing importance in civil engineering and building structures.

In this paper an optimization problem is formulated and solved in which the amount of fracture energy absorbed at failure is considered as an objective function and its maximum is sought. In several applications this may be considered, with good practical reason, as an appropriate measure of the material quality.

The fibre direction is the only variable and its optimal value represents the solution, which is discussed for various sets of parameters. The role of the fibre orientation was considered by Morton who showed in [2] that the trivial case of all fibres parallel to the principal tensile stresses is not an optimal solution. This conclusion was corroborated by photoelastic observations and other tests published by Hing and Groves [3], Harris *et al.* [4] and Morton and Groves [5].

The calculation of the fracture energy based on simplifying assumptions was proposed and experimentally verified on notched steel fibres reinforced cement beams tested by Brandt [6]. Similar formulae were used later in [7] for an optimization problem which is further developed below.

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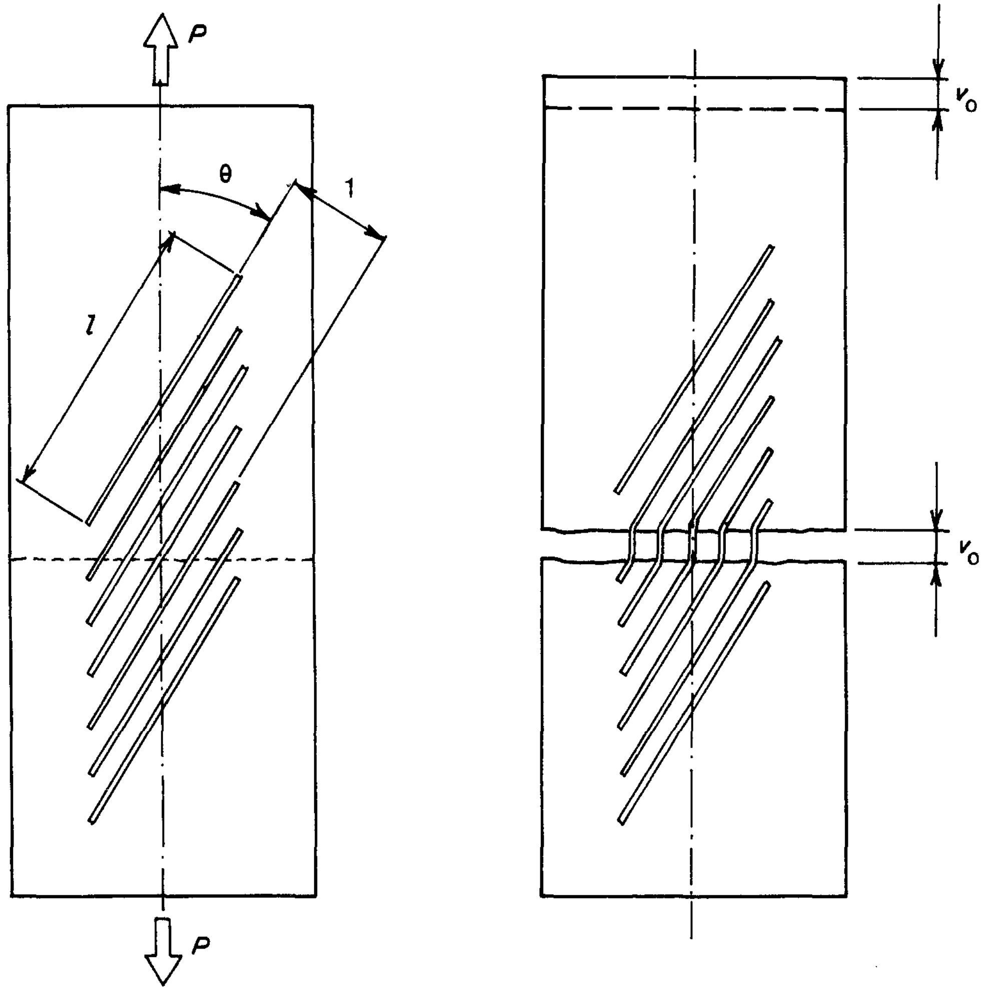

Figure 1 Fibre reinforced concrete element subject to tension before and after the crack opening.

2. Optimization of the set of parallel fibres orientation in an element subject to axial tension

An element of steel fibre reinforced cement subject to axial tension is considered. The element may form a part of a structure or of a layer for which tensile forces are of major importance. For structural elements subject to other loading states, the behaviour under tension may also be considered as a severe but appropriate measure of utility.

The reinforcement is composed of a single system of parallel short steel fibres and θ is its angle with respect to the direction of the tensile loading. It is assumed that in a neighbouring layer of the reinforcement the respective angle is $-\theta$. The reinforcement being symmetric, only one form of rupture is admissible: a single crack perpendicular to the load direction (see Fig. 1). The problem of multiple cracking should be considered separately.

The objective function is the amount of energy absorbed to produce a crack in the element and to open it to a certain width, v_0. This crack opening is therefore considered as the final fracture of the element.

The following components of the fracture energy are taken into account:

(a) debonding of matrix from the fibres which cross the crack,

(b) pulling of debonded fibres out of the matrix, the fibre displacement is equal to the crack width,

(c) passing of the fibres across the crack.

To simplify formulation of the problem and its solution, it is assumed that these energy

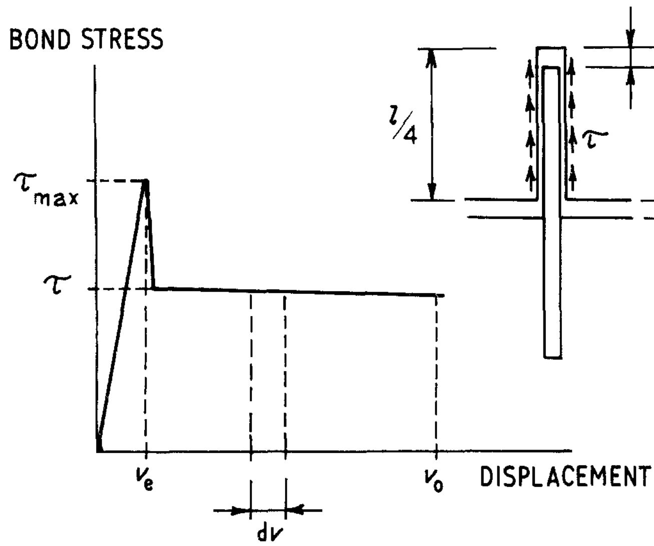

Figure 2 Bond stress as function of displacement in a pull-out test.

components are mutually independent. It is supposed also that the following components may be neglected:

(i) creation of a new surface of the crack in the matrix, because it is independent of the influence of the fibres,

(ii) strain elastic energy in the element, because it is absorbed before the crack appears and its amount depends mostly on the volume of the element itself.

By this way a simplified mechanism is created which is described below analytically. This mechanism may be modified in the future, according to new concepts or test results, which are not available at present.

In all considerations here, the properties of cement matrices and steel fibres are taken into account. It is possible, however, to generalize the main concept of the presented problem for larger groups of materials, e.g. plaster or polymer based matrices with glass or polypropylene fibres.

3. Calculations of the components of the fracture energy

3.1. Energy due to debonding of fibres from the matrix

The energy is proportional to the area corresponding to the elastic displacement in a typical simplified diagram obtained in a pull-out test, shown in Fig. 2. It is expressed by the formula

$$W_1 = N_0 \cos^2\theta \, \frac{l\pi D}{8} \, \tau_{\max} \, v_e \qquad (1)$$

where N_0 is the number of fibres per unit cross-section perpendicular to the fibres, D and l are the diameter and the length of a fibre, the maximum bond stress $\tau_{\max}$ and crack opening v_e corresponding to elastic displacement of fibre with respect to matrix are indicated in Fig. 2. $N_0 \cos\theta$ is the number of fibres per unit area of crack surface. Another $\cos\theta$ in the formula comes from the projection of the displacement on the fibre direction. The force in a fibre is: $\frac{1}{4} l\pi D\tau_{\max}$.

Here, the pull-out length is assumed to be equal to $l/4$ as an average of all fibre lengths between 0 and $l/2$.

3.2. Energy due to pulling fibres out of the matrix

The pulling-out of fibres against the interfacial friction τ along the fibre length is considered. The friction appears in the cylindrical crack around the deboning fibre (Fig. 2). The energy is proportional to the area under the second part of the diagram, between the displacements v_e and v_0. The elementary energy, $\mathrm{d}W_2$, is expressed by $\mathrm{d}W_2 = \pi\tau D(l/4 - v)\,\mathrm{d}v$. After multiplication by the number of fibres and after integration, the final formula is

$$\begin{aligned} W_2 &= N_0 \cos^2\theta \int_{v_e}^{v_0} D\pi\tau \left(\frac{l}{4} - v\right) \mathrm{d}v \\ &= N_0 \cos^2\theta D\pi\tau \\ &\quad \times \left[\frac{l}{4}(v_0 - v_e) - \tfrac{1}{2}(v_0^2 - v_e^2)\right] \end{aligned} \qquad (2)$$

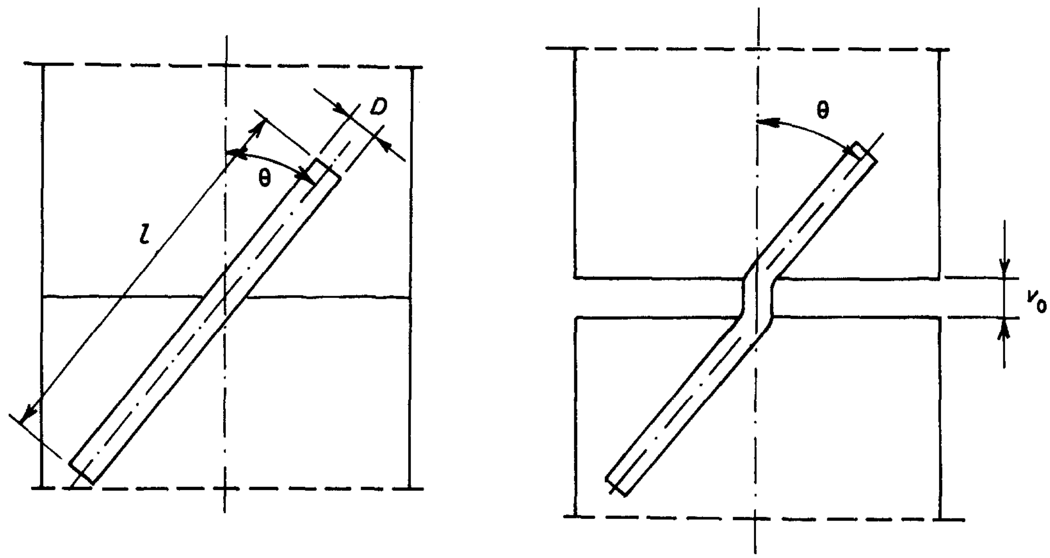

Figure 3 Plastic deformation of a fibre passing across a crack at an angle θ.

The concept of this formula was taken from [4] and [8] and it was developed in [6].

3.3. Energy due to fibres passing across a crack

Three main phenomena are considered separately here:

(a) plastic deformation of the fibres due to shear stress,

(b) crushing of the matrix in compression near the exit points of the fibres,

(c) complementary friction between the fibres and the matrix due to local compression at the bend.

These phenomena are based on proposals published by Aveston and Kelly [9].

Plastic deformation of the fibre is shown schematically in Fig. 3. According to the proposal in [8] the energy necessary to deform a fibre is proportional to the fibre volume between the crack edges, to the shear yield stress τ_f of steel and to the angle θ. Therefore the formula is

$$W_3 = N_0 \cos\theta \frac{\pi D^2}{4} \theta v_0 \tau_f \qquad (3)$$

where $(\pi/4)D^2 v_0$ is the fibre volume subject to yielding, τ_f is considered to be equal to a half of the tensile yield stress, f_f and $N_0 \cos\theta$ is the number of fibres.

Crushing of the matrix which is locally compressed may be represented by Fig. 4. The energy is approximately proportional to the matrix strength and to the volume of the crushed matrix on both sides of the crack. The radius r_0 may be calculated from the approximate relation

$$r_0 = D\alpha \frac{f_f}{f_m}\left(\frac{\cos^2\theta}{\sin\theta}\right)^{1/2}$$

where f_f and f_m are the fibre and the matrix strength and α is the coefficient which should modify the value of r_0 to adjust it to the results of observations. Then:

$$\text{area } S = \tfrac{1}{2} r_0^2 (\tan\theta - \theta)$$

$$\text{volume } V = \tfrac{1}{2} D^3 \left(\alpha \frac{f_f}{f_m}\right)^2 \times \frac{\cos^2\theta}{\sin\theta}(\tan\theta - \theta)\eta.$$

Finally, the formula for the corresponding energy component is

$$W_4 = N_0 \left(\alpha \frac{f_f}{f_m}\right)^2 \eta f_m D^3 \times \left(\cos^2\theta - \theta \frac{\cos^3\theta}{\sin\theta}\right) \qquad (4)$$

taking into account two crushed zones of matrix on both sides of the crack. The coefficient η should correct the approximate formula for the matrix volume.

The complementary friction is produced by the force **BA** shown in Fig. 5 and may be approximately calculated as $2P \sin\theta/2$, where P,

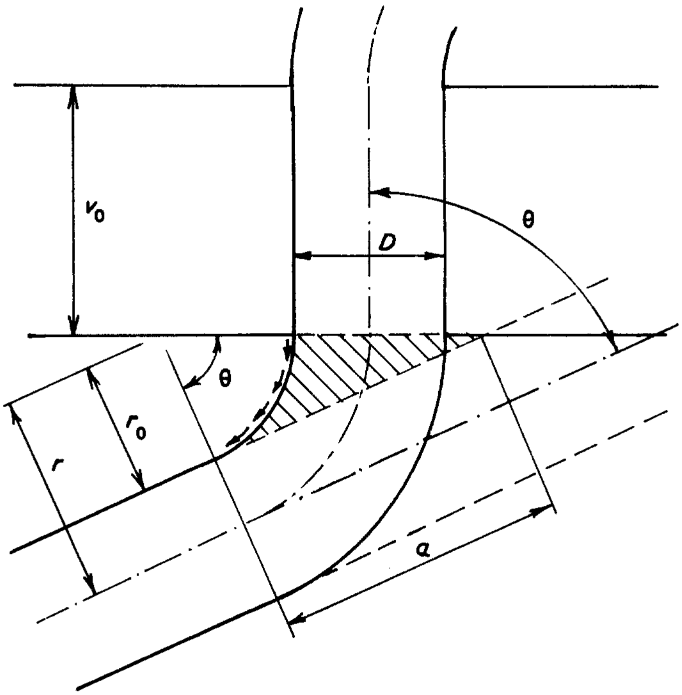

Figure 4 Crushing the matrix compressed by a fibre.

the force which pulls out the fibre from the matrix is composed of two parts

$$P = P_0 + \Delta P.$$

Force P_0 is already taken into account in W_2 and the additional force ΔP may be expressed by

$$\Delta P = \frac{\pi}{4} l\pi D\phi 2 \sin\frac{\theta}{2}$$

where ϕ is the friction coefficient between fibre and matrix.

Therefore the energy is

$$W_5 = N_0 \pi l \tau D \phi v_0 \sin\frac{\theta}{2}\cos\theta \qquad (5)$$

where both sides of a crack are considered.

Total fracture energy is, according to the initial assumptions, equal to a sum of these five components:

$$W_\theta^{\mathrm{ID}} = \sum_1^5 W_i, \qquad (i = 1, 2, \ldots, 5). \quad (6)$$

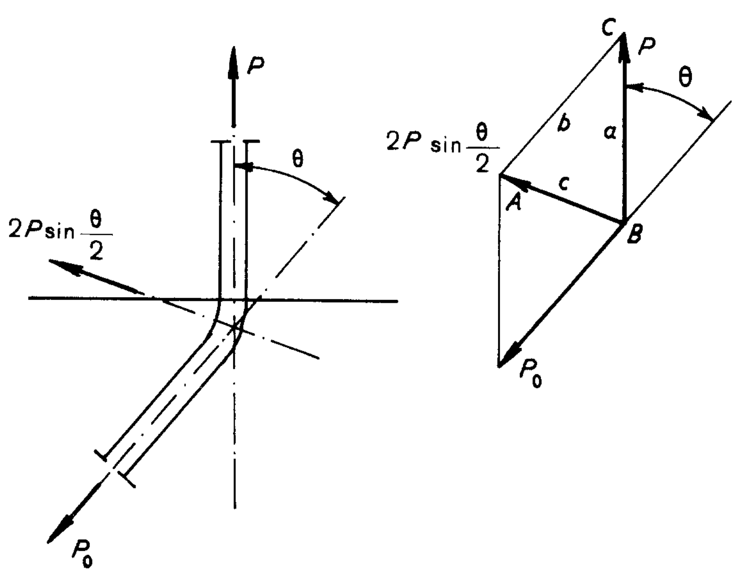

Figure 5 Additional force at the bend of a fibre.

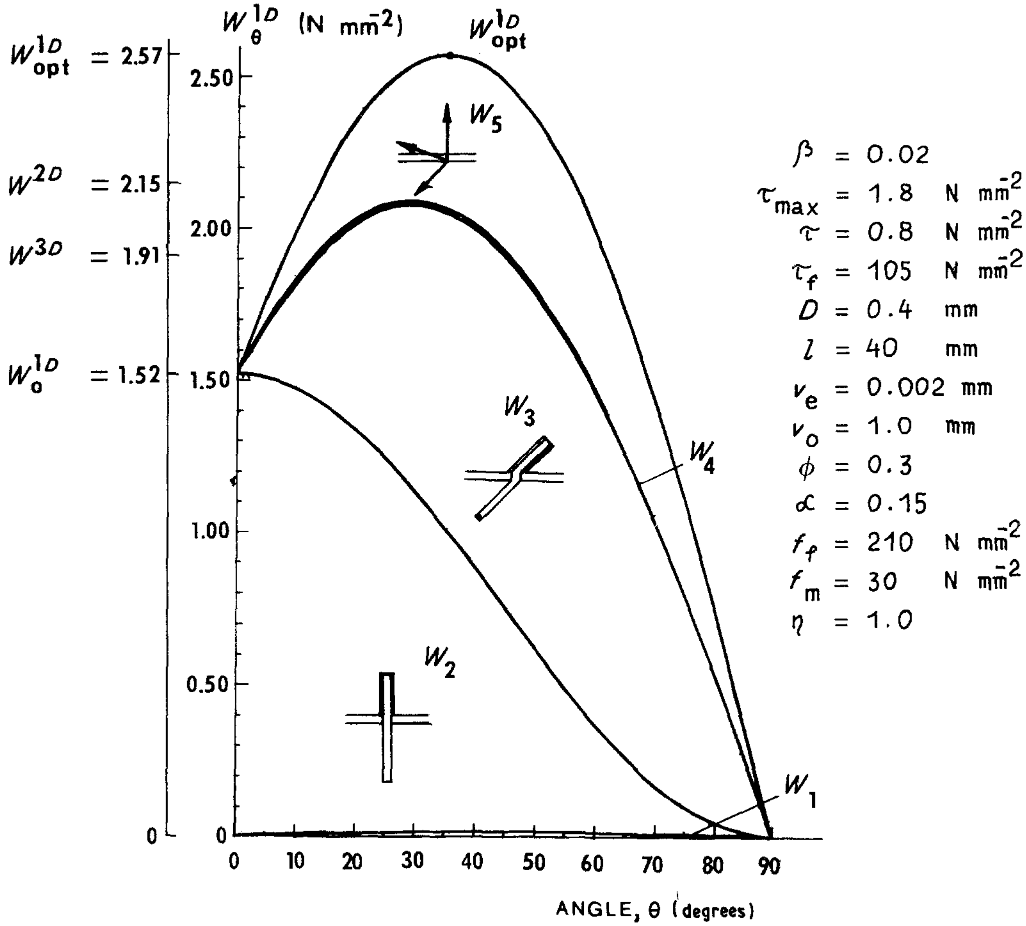

Figure 6 Diagram of the fracture energy W^{1D} as a function of the angle θ. Example 1.

After substituting all the previously obtained relations into Equation 6, the total energy is expressed by the following formula

$$W_\theta^{1D} = N_0 D l v_e \frac{\pi}{8} \tau_{max} \cos^2\theta + N_0 D\pi\tau\left[\frac{l}{4}(v_0 - v_e) - \tfrac{1}{2}(v_0^2 - v_e^2)\right]\cos^2\theta + N_0 D^2 v_0 \tau_f \frac{\pi}{4}\theta\cos\theta + N_0 D^3 f_m \left(\alpha\frac{f_f}{f_m}\right)^2 \eta \times \left(\cos^2\theta - \theta\frac{\cos^3\theta}{\sin\theta}\right) + N_0 D l v_0 \pi\phi \sin\frac{\theta}{2}\cos\theta \quad (7)$$

The indices 1D and θ recall that the energy is in the element reinforced with a 1D system of parallel fibres inclined at an angle θ to the direction of tensile load. The number of fibres is given by the formula

$$N_0 = \frac{4\beta}{\pi D^2} \quad (8)$$

where β is the volume fibre content.

For two examples of sets of parameters, the values of W_θ^{1D} as functions of θ are calculated from Equation 7 and the results are shown in Figs. 6 and 7.

For the calculations all numerical values of the parameters are taken from tests of steel fibre reinforced cements. These values are indicated in the legends of the figures.

Among other conclusions it may be observed that the function W_θ^{1D} has an extremum and that it corresponds to a maximum. The influence of different energy components is strongly dependent on the values of particular parameters.

The necessary condition for the energy extremum is

$$\frac{\partial W_\theta^{1D}}{\partial\theta} = 0 \quad (9)$$

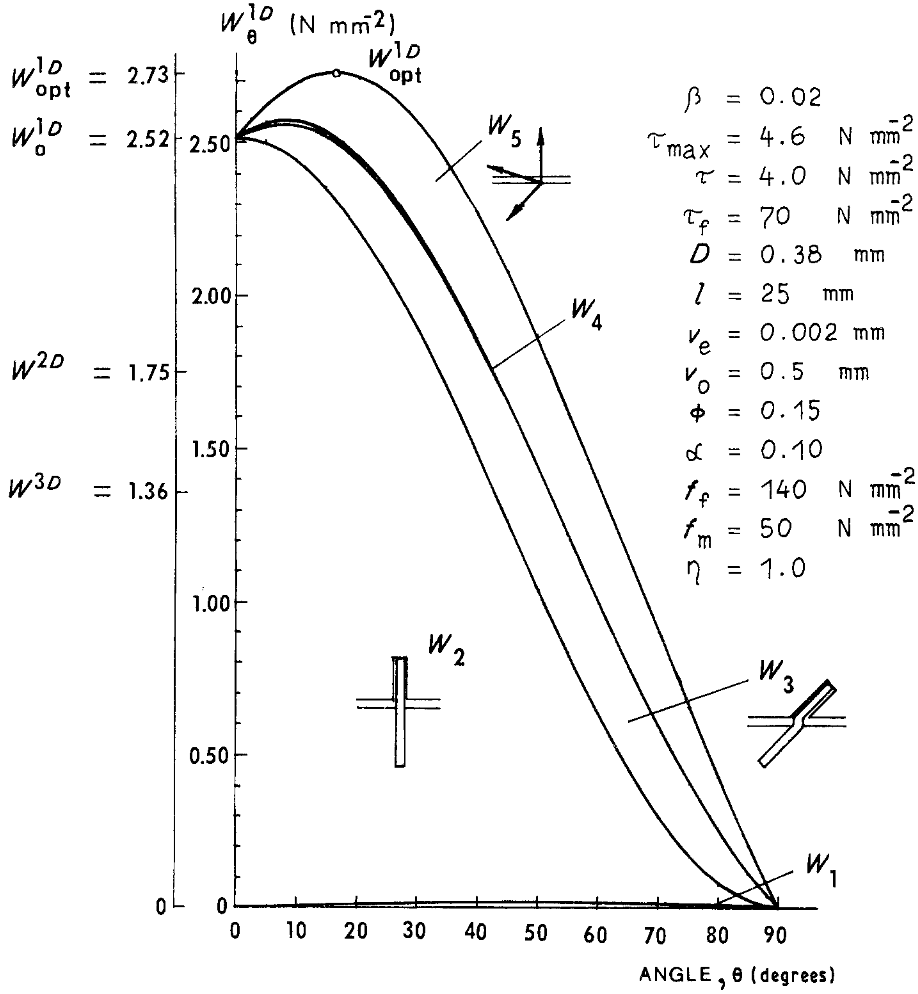

Figure 7 Diagram of the fracture energy W^{1D} as a function of the angle θ. Example 2.

After substitution of Equation 7 into Equation 9 two conditions are obtained from Equation 7

$$\cos\theta = 0, \tag{10}$$

or

$$-\frac{\pi}{4}Dl\tau_{max}v_e\sin\theta$$
$$-\pi D\tau\left[\frac{l}{2}(v_0 - v_e) - (v_0^2 - v_e^2)\right]\sin\theta$$
$$+\frac{\pi}{4}D^2\tau_f v_0(1 - \theta\tan\theta)$$
$$+D^3 f_m\left(\alpha\frac{f_f}{f_m}\right)^2\eta$$
$$\times\{[(3+\cot^2\theta)\theta - \cot\theta]\cos\theta - 2\sin\theta\}$$
$$+\pi Dl\tau\phi v_0\left(\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\tan\theta\right) = 0. \tag{11}$$

Equation 10 only gives a trivial solution: $\theta = \pi/2$ which corresponds obviously to a minimum of W_θ^{1D}, because all of the fibres are not acting across the crack. The solution of Equation 11 was obtained by numerical methods for several sets of parameters. For the example shown in Fig. 6 the optimal value, θ_{opt}, is 35°03′52″ and for the example in Fig. 7 θ_{opt} = 15°47′24″. The optimal angle, θ_{opt}, is considered below as a function of various parameters.

4. Optimal value of the angle θ for 1D systems of fibres

Equation 11 may be used to calculate optimal values of the angle θ designed by θ_{opt}, as a function of various parameters. A few examples are given here for sets of parameters taken from tests of steel fibre reinforced cement based composites.

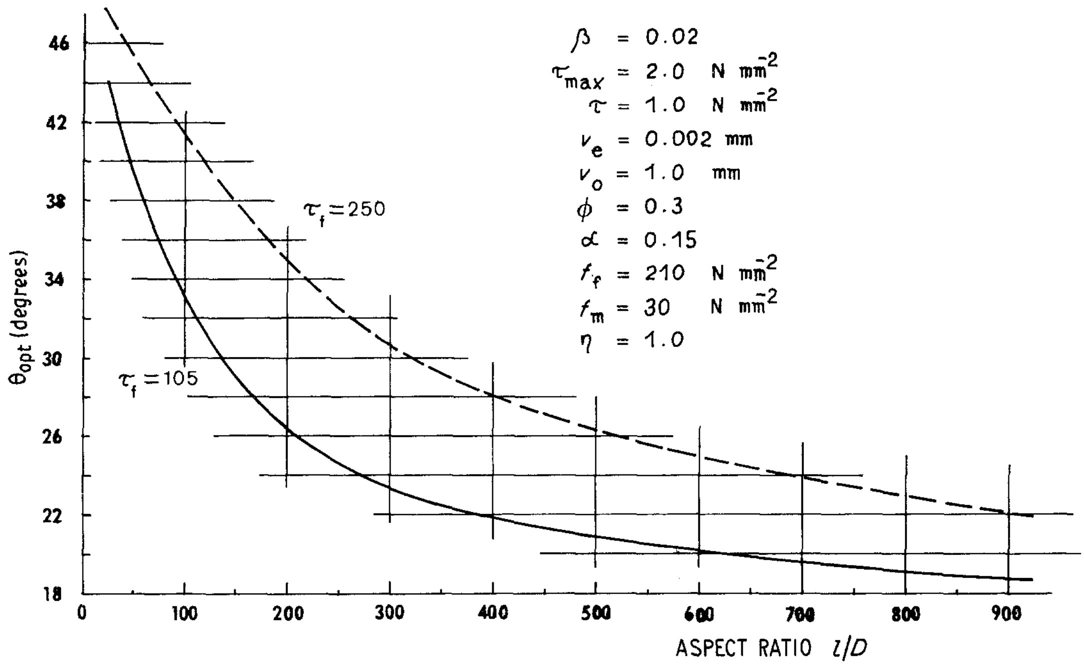

Figure 8 Optimal angle θ_{opt} as a function of the aspect ratio l/D of a single fibre.

For an aspect ratio, l/D, characterizing the shape of a single fibre, varying from 33 to 900, the angle θ_{opt} decreases considerably (see Fig. 8). This means that for short and thick fibres θ_{opt} is equal to about 43 to 46° and for long and thin fibres the angle θ_{opt} decreases to about 18 to 21°. Higher values correspond to stronger fibres with $\tau_f = 250\,\text{N}\,\text{mm}^{-2}$ and lower values to weaker fibres with $\tau_f = 105\,\text{N}\,\text{mm}^{-2}$. It is assumed here that the relation exists between the shear yield stress τ_f and the tensile strength of steel f_f is

$$f_f = 2\tau_f. \qquad (12)$$

The quality of the bond between fibres and matrix also has an important influence on the angle θ_{opt}. Its value decreases when the bond stresses increase, as it is shown in Fig. 9. Values of τ_{max} and τ equal to 1.4 and $0.4\,\text{N}\,\text{mm}^{-2}$ respectively correspond to plain fibres. The value of 2.6 and $1.6\,\text{N}\,\text{mm}^{-2}$ may be considered as appropriate for slightly indented fibres. The influence of the fibre shearing strength τ_f is also observed: stronger fibres require higher values of θ_{opt} as it is shown in Fig. 10. This conclusion is confirmed also by results obtained previously shown in Fig. 8.

The influence of the crack width, v_0, defined as a fracture of the element, is rather small as it is shown in Fig. 11: for v_0 varying from 0.1 to 1 mm the angle θ_{opt} varies only from about 34°30′ to 32°40′. It may be verified also that the coefficients α and ϕ do not influence considerably the angle θ_{opt}.

5. Energy accumulated in elements with 2D and 3D systems of fibres

The results of optimization obtained above may be compared with energies absorbed in elements reinforced by fibres distributed at random in planar (2D) or in spatial (3D) systems. Corresponding formulae may be obtained by appropriate integration of Equation 7 and the following expressions are found

$$\begin{aligned} W^{2D} &= 0.202642 \frac{\beta}{D} l v_e \tau_{max} \\ &+ 0.405285 \frac{\beta}{D} \tau \left[l(v_0 - v_e) - 2(v_0^2 - v_e^2)\right] \\ &+ 0.5\beta v_0 \tau_f + 0.086910 D\beta f_m \left(\alpha \frac{f_f}{f_m}\right)^2 \eta \\ &+ 0.949641 \frac{\beta}{D} l v_0 \tau \phi \end{aligned} \qquad (13)$$

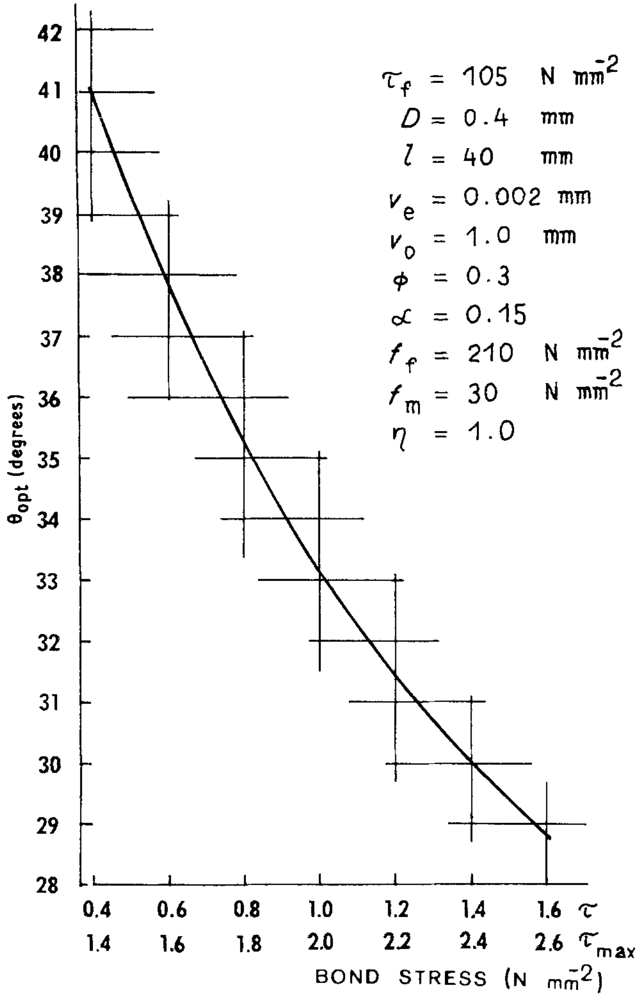

Figure 9 Optimal angle θ_{opt} as a function of the band characteristics τ and τ_{max}.

and

$$W^{3D} = 0.125\frac{\beta}{D}lv_e\tau_{max} + 0.25\frac{\beta}{D}\tau[l(v_0 - v_e) - 2(v_0^2 - v_e^2)] + 0.5\beta v_0\tau_f + 0.084767\beta D\eta\left(\alpha\frac{f_f}{f_m}\right)^2 f_m + 0.942809\frac{\beta}{D}lv_0\tau\phi. \quad (14)$$

Numerical values of energies calculated from Equations 13 and 14 for two different sets of parameters are shown in Figs. 6 and 7. It may be observed that in the case shown in Fig. 6 the element with dispersed fibres of 2D or 3D systems can absorb a considerably greater amount of energy than the element with all the fibres parallel to the direction of principal tensile stress. In the other case, represented in Fig. 7, parallel fibres ensure higher absorbed energy than any random system. The difference consists of a different selection of main parameters, such as qualities of fibres and matrices.

There are few test results published until now with enough data available to calculate the absorbed energy and to confirm the above analytical results. Only in the tests published by Kasperkiewicz [10] are necessary data given from which it may be calculated that

$$W_0^{1D} \cong 3W^{2D}.$$

This means that the parallel fibre system with the angle $\theta = 0$ gives 3 times higher energy absorbed before the fracture than the fibres dispersed at random (2D). By these tests, the model presented in Fig. 7 is confirmed. Further experimental work in this field is planned to calculate energy as a function of the angle θ and

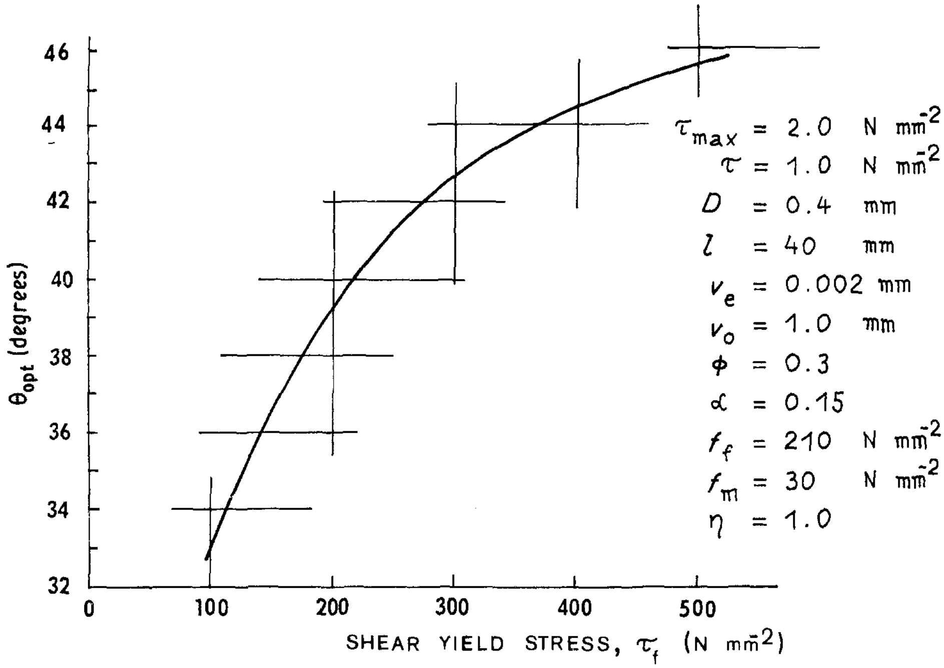

Figure 10 Optimal angle θ_{opt} as a function of the shearing yield stress τ_f of fibres.

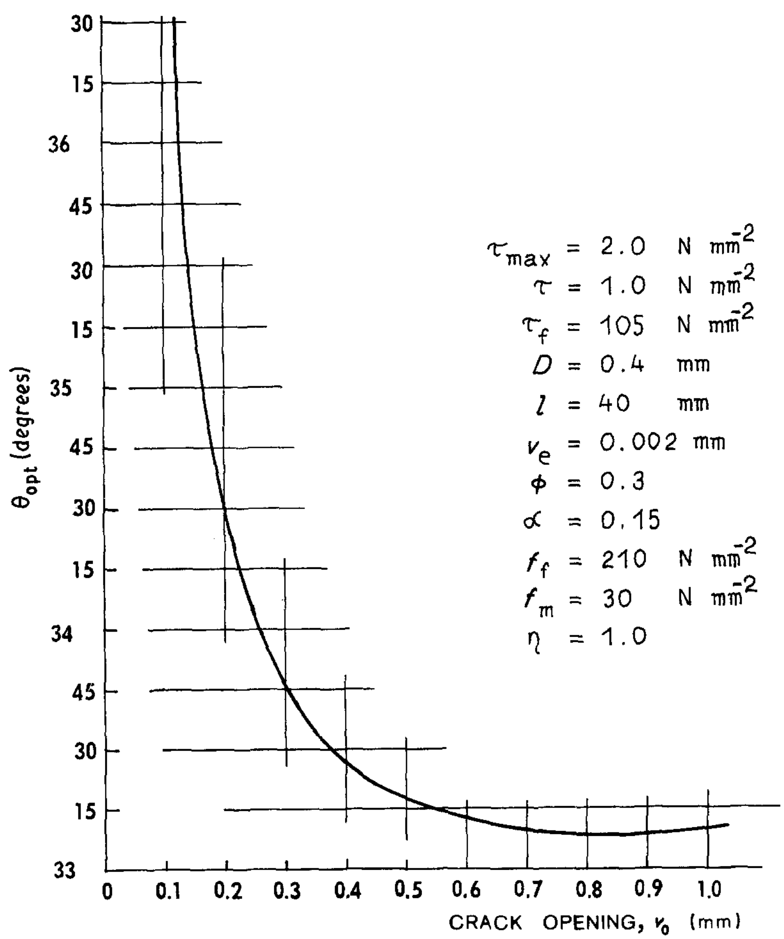

Figure 11 Optimal angle θ_{opt} as a function of the final crack opening v_0.

to check the above shown analytically obtained diagrams.

6. Conclusions

In the above described analytical considerations the optimal values of the angle θ are determined and discussed as functions of several particular parameters. The numerical values of θ_{opt} are obtained in several examples. The values of corresponding fracture energies in elements, reinforced with parallel fibres and with random fibres in 2D and 3D systems, are compared.

It appears, for example, that for certain fibre and matrix properties, the fibres at angle $\theta = 0$ give a smaller amount of absorbed fracture energy than the random 2D or 3D systems of fibres.

The obtained solutions give valuable information about the influence of different parameters on the energy absorbed by the element before the fracture and about the correct choice of the reinforcement system.

The proposed simplified model of an element subject to tension yields rather coherent results in optimization problems. Further studies are directed at considering a more realistic and complicated model of the material behaviour and at taking into account other parameters and design variables. Among others the influence of the multiple cracking will be considered. Experimental investigations should also be undertaken to give enough data for general checking of the proposed model.

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